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Optimal Mass Transport for Statistical Estimation, Image Analysis, Information Geometry, and Control

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We developed several new directions in the theory and applications of Optimal Mass Transport (OMT). OMT has its origins in civil engineering (Monge 1781) and economics (Kantorovich 1942), but in recent years has increasingly impacted a large number of other fields (probability theory, partial differential equations, physics, meteorology). We have addressed computational aspects of the problem and the need for further expanding the arsenal of computational tools. We considered a wide range of generalizations and insights for the purpose of tackling problems of AFOSR interest. These include matrix-valued statistics and fusion of information, optical flow, controlled active vision, tracking and dynamic textures.

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PI: Tryphon T. Georgiou
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Abstract

We developed several new directions in the theory and applications of Optimal Mass Transport (OMT). OMT has its origins in civil engineering (Monge 1781) and economics (Kantorovich 1942), but in recent years has increasingly impacted a large number of other fields (probability theory, partial differential equations, physics, meteorology). We have addressed computational aspects of the problem and the need for further expanding the arsenal of computational tools. We considered a wide range of generalizations and insights for the purpose of tackling problems of AFOSR interest. These include matrix-valued statistics and fusion of information, optical flow, controlled active vision, tracking and dynamic textures.

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Status/Progress

We have accomplished the following in our program and proposed research:

(i) ***Computational Tools for Optimal Mass Transport (OMT)***: We developed a number of tools allowing us to solve several problems, including the construction of geodesics, computation of metric distances, and transportation means. Such constructions are motivated by a variety of engineering applications. Further, we have exploited the beautiful connection between the Boltzmann entropy and the heat equation. The latter arises as the steepest descent when maximizing the Boltzmann entropy "potential" in the Riemannian metric inherited on the space of probability densities via OMT. The rate of ascent, since entropy increases, is given by the Fisher information metric. The same paradigm has been used to recover/generate other gradient flows (PDEs) and thereby link via suitable information potentials to corresponding metrics for distributions.

(ii) ***Power spectra***: Our work on high resolution signal analysis has led to a number of novel notions of distance between power spectra with applications to prediction theory. We have investigated the topic of matrix-valued power spectra as it relates to several DoD interests. In this regard, prediction theory is juxtaposed with developments in quantum information theory, creating a synergy of methodologies for signal analysis.

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(iii) ***Tracking and Mesh Generation***: We have continued our work in visual tracking and controlled active vision. Mass transport is being used as a comparison metric on shapes as part of a feedback loop for tracking in conjunction with statistical filtering. Further, for various problems in computational fluid dynamics, biomechanics, and CAD, we have developed novel techniques based on OMT that may be employed for the automatic generation of hexahedral meshes for three dimensional volumes.

(iv) ***Optimal transport on networks***: We have made significant advances on formulating and solving optimal transport problems on discrete spaces (networks) while ensuring robustness of the transportation plan. This work makes contact with a probabilistic formalism of bridging two probability distributions along path of a random walker that displays the two given marginals. The development builds on the theory of the so-called Schrödinger bridges and opens up new directions for investigating geometries of mass and probability distributions.

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Ivan Kolesov (Ph.D. student-graduated; now working in a start-up)
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Other Related Patents Granted During previous AFOSR Projects

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(b) “Curvature Based System for the Segmentation and Analysis of Cardiac Magnetic Resonance Imagery,” (co-inventors Allen Tannenbaum and Anthony Yezzi), U.S. Patent Number 6,535,623 B1, issued March 18, 2003.

(c) ”Bayesian Methods for Noise Reduction in Image Processing,” (co-inventors Allen Tannenbaum and Ben Fitzpatrick), U.S. Patent Number 7,813,581, issued October 12, 2010.

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Transitions

Performer: Allen Tannenbaum, Stony Brook University, (404) 394-3224.
Customer: Polaris/Lockheed Martin, Art Lompado/Moses Chan.
Result: Tracking algorithms for multiple unresolved targets in noisy environments.
Application/Military Utility: Detection and tracking of maneuvering missiles.
Performer: Allen Tannenbaum, Stony Brook University, (404) 394-3224.
Customer: UTRC, Amit Surana.
Result: Video analytics algorithms.
Application/Military Utility: Studying anomalous behaviors in groups in people.

DISTRIBUTION A: Distribution approved for public release.

1 Introduction

Optimal mass transport is a major research area with applications for numerous disciplines including econometrics, fluid dynamics, automatic control, transportation, statistical physics, shape optimization, expert systems, and meteorology [52, 68]. The problem was first formulated by the civil engineer Gaspar Monge in 1781, and concerned finding the optimal way, in the sense of minimal transportation cost, of moving a pile of soil from one site to another. Much later the problem was extensively analyzed by the Soviet mathematician Kantorovich [35] with a focus on economic resource allocation, and so is now known as the Monge–Kantorovich (MK) or optimal mass transport (OMT) problem.

A major problem that elucidates how OMT is employed is image registration [28]. Since this appears in many practical systems, tracking, and vision applications, we will briefly explain how OMT may be used to treat this problem. Registration is the process of establishing a common geometric reference frame between two or more data sets obtained by possibly different imaging modalities and at different times. Registration typically proceeds in several steps. First, a measure of similarity between the data sets is established, so that one can quantify how close an image is from another after transformations are applied. Such a measure may include the similarity between pixel intensity values, as well as the proximity of predefined image features such as implanted fiducials, anatomical landmarks, surface contours, and ridge lines. Then the transformation that maximizes the similarity between the transformed images is found. Many times this transformation is given as the solution of an optimization problem where the transformations to be considered are constrained to be members of a predetermined class. Lastly, once an optimal transformation is obtained, it is used to fuse the image data sets. Registration has a huge literature devoted to it with numerous approaches ranging from statistical to computational fluid dynamics to various types of warping methodologies; see [66, 70]. One way of defining density is via "intensity," and in such a case the method explicated in this proposal can be considered an intensity-driven one. The method we devised as part of our AFOSR funded research, is also in the class of warping strategies based on continuum and fluid mechanics, in which one tries to use properties of elastic materials to determine the deformation. Here one defines a (typically quadratic) cost functional that penalizes the mismatch between the deforming template and target. A key fact that will be employed when we describe a Riemannian metric on the space of probability densisites is that the optimal warping map of OMT may be regarded as the velocity vector field which minimizes a standard energy integral subject to the Euler continuity (mass preservation) equation [4]. Thus, the theory of OMT allows a natural geometry on the space of distributions and suitable warping maps and geodesics which establish correspondence between distributions and may be used suitably in applications.

With this background, we have investigated the following problems in our just completed AFOSR research program:

(i) ***Riemannian Metrics and Gradient Flows***: There is a beautiful connection between the Boltzmann entropy and the heat equation. The latter arises as the steepest descent when maximizing the Boltzmann entropy "potential" in the Riemannian metric inherited by the

OMT. The rate of ascent, since entropy increases, is given by the Fisher information metric. The same paradigm can be used to recover/generate other gradient flows (PDEs) and thereby link via suitable information potentials to corresponding metrics for distributions. In particular, we recover the affine invariant heat equation and draw connections with metrics on power spectra. This topic is primarily of theoretical interest with potentially important insights into the qualities of various alternative metrizations for the space of distributions.

(ii) ***Optical Flow, Tracking, and Mesh Generation***: Optical flow is a key problem in controlled active vision and thus in visual tracking. The optical flow field is defined as the velocity vector field of apparent motion of brightness patterns in a sequence of images. There have been many methods proposed for its computation. We have shown that show that ideas from optimal mass transport are ideal for the computation of the optical flow field for scenarios involving *dynamic textures*, that is, for objects that have internal dynamics such as fire and smoke. Further, for various problems in computational fluid dynamics, biomechanics, and CAD, we have shown that techniques from OMT may be employed for the automatic generation of hexahedral meshes for three dimensional volumes.

2 Summary of Work

We summarize some of the key results developed as part of our AFOSR research program. Optimal mass transport (OMT) and the mathematics that were spawn from the Monge-Kantorovich problem have impacted a number of fields including probability theory, statistics, physics, the atmospheric sciences, economics, and functional analysis; e.g., see [2, 4, 52]. Our research focused around around problems in information fusion and control, and image registration.

2.1 Geometry of optimal mass transport

Consider once again the OMT problem. In our program, we studied, in particular, costs of the form $\rho(u, x) = |u - x|^p$ ($p \geq 1$), giving rise to the L^p *Kantorovich–Wasserstein metric*

$$d_p(\mu_0, \mu_1)^p := \inf_{u \in MP} \int \mu_0(x)|u(x) - x|^p \, dx. \tag{1}$$

Thus, an *optimal MP-map*, when it exists, is one which minimizes the above integral. The integral represents a cost on the distance the map u moves each bit of material, weighted by the respective mass.

The case $p = 2$ has been extensively studied in recent years. A fundamental result by Yann Brenier [10] is that *there is a unique optimal* $u \in MP$ *transporting* μ_0 *to* μ_1*, and that this* u *is characterized as the gradient of a convex function* w*,* i.e., $u = \nabla g$ where g can be thought as a *convex potential*. The geometric significance of this can be traced to the fact that for a transference plan to be optimal, there should be no "crossing" of paths that individual specs of mass take. This insight [13, 40] forces the graph of the optimal plan to have a certain *cyclically monotone* property, which then, by a theorem of Rockafeller [59], implies that it is the (sub-)differential of a convex

function. The novelty of this result is that very much like the Riemann mapping theorem in the plane, OMT singles out particular maps with preferred geometry.

It is interesting to speculate whether a similar geometric insight is relevant in optimal multivariable couplings amongst more than one distribution. Such problems will be motivated later on in the context of fusion of information from various sources/sensors. Recent works [60, 24] motivate analogous questions but rather from an economics perspective. Thus, a question of potential great relevance is to study in a similar manner (**research direction**) the geometric properties of solutions to the multi-transport problem $\min_\mu \sum_{i=1}^n d_p(\mu, \mu_i)$ for a given set of distributions μ_i, and develop computational tools for this problem.

2.1.1 Optimal mass transport as an optimal control problem

The Monge-Kantorovich problem for $p = 2$ may be formulated as follows [4]. Consider

$$\inf \int \int_0^1 \mu(t,x) \|\nabla g(t,x)\|^2 \, dt \, dx \tag{2}$$

over all time varying densities μ and functions g satisfying

$$\begin{aligned} \frac{\partial \mu}{\partial t} + \text{div}(\mu \nabla g) &= 0, \\ \mu(0,\cdot) &= \mu_0, \ \mu(1,\cdot) = \mu_1. \end{aligned} \tag{3}$$

The integrand in (2) may be thought to represent kinetic energy with $u = \nabla g$ representing velocity. Thus, (2) is an "action" integral in the way understood in the physics literature. One may then show that the infimum is attained for some μ_{min} and g_{min}; accordingly we set $u_{min} = \nabla g_{min}$. Further, define the flow

$$X(x,t) = x + t(u_{min}(x) - x).$$

Note that when $t = 0$, X is the identity map and when $t = 1$, it is the solution u_{min} to the Monge–Kantorovich problem. This analysis provides appropriate justification for using (2.1.1) to *define* a continuous (nonlinear) warping map X between the densities μ_0 and μ_1. Besides the relevance of such a warping for applications such as image registration [28], voice morphing [33], etc., the analysis above provides a Riemannian structure on the space of distributions that we take up next.

2.1.2 Riemannian structure of density functions

Define the space of density functions as

$$\mathcal{C} := \{\mu \geq 0 : \int \mu = 1\}.$$

The tangent space at a given "point" μ may be identified with

$$T_\mu \mathcal{C} \cong \{v : \int v = 0\}.$$

Thus, inspired by the Benamou-Brenier framework [4], given two "points" $\mu_0, \mu_1 \in \mathcal{C}$, the geodesic distance is:

$$\begin{aligned}\inf_{\mu,g}\{\int\int_0^1 \mu(t,x)\|\nabla g(t,x)\|^2\,dt\,dx \\ \text{subject to } \frac{\partial\mu}{\partial t} + \text{div}(\mu\nabla g) = 0, \\ \mu(0,\cdot) = \mu_0,\ \mu(1,\cdot) = \mu_1\}\end{aligned} \tag{4}$$

In other words, we look at all curves in $\mathcal{C}$ connecting μ_0 and μ_1, and take the shortest one with respect to the Wasserstein metric. This leads us to give $\mathcal{C}$ a Riemannian structure, which will induce the Wasserstein distance. This idea is in fact due to Jordan *et al.* [34]. Namely, under suitable assumptions on differentiability for $\mu \in \mathcal{C}$, and $v \in T_\mu\mathcal{C}$, one solves the Poisson equation

$$v = -\text{div}(\mu\nabla g). \tag{5}$$

This allows us to identify the tangent space with functions g up to additive constant. Thus, for any given v we denote the solution of (5) by g_v. Then given, $v_1, v_2 \in T_\mu\mathcal{C}$, we can define the inner product

$$\langle v_1, v_2\rangle_\mu := \int \mu\nabla g_{v_1}\cdot\nabla g_{v_2}. \tag{6}$$

An integration by parts argument, shows that this inner product will exactly induce the Wasserstein distance given by Equation (4). It is very suggestive to also note that

$$\begin{aligned}\langle v, v\rangle_\mu &= \int \mu\nabla g_v\cdot\nabla g_v = -\int g_v\text{div}(\mu\nabla g_v) \text{ (integration by parts)} \\ &= \int v g_v.\end{aligned} \tag{7}$$

Several interesting questions were explored explored in detail. For instance, typically, action integrals $\int(T - V)$ in physics have both a term corresponding to the kinetic energy T and one corresponding to a potential V. In our AFOSR work, in several publications, we have studied the implication of a potential term to the action integral (2) and how this affected the induced geometry.

2.2 Riemannian metrics & gradient flows

The availability of a natural metric structure on the space $\mathcal{C}$ of distributions suggested a range of interesting theoretical questions that were investigated in our program. This line of research has led to the the rather deep and surprising fact that gradient flows of the Boltzmann entropy in the geometry of Wasserstein-Kantorovich metric give rise to the ***heat equation*** [34]. A similar approach gives rise to the ***affine-invariant nonlinear heat equation*** (co-discovered by one of the PI's (AT)) which has been of great significance and popularity in image processing [61, 62], and connections are drawn with our recent work on a differential-geometric structure based on ***optimal prediction theory*** for spectral density functions [21]. Thus, the geometry of OMT links together the

Boltzmann entropy, the heat equation and, as we will see in the next section, the Fisher information metric. In a similar way, the use of an information potential other than the Boltzmann entropy can give rise to alternative gradient flows and metrics that relate to affine-invariance [61], prediction theory [21], flow in porous media and many other important paradigms in physics [68]. In our AFOSR research, we have elucidated the relationship between mass transport, conservation laws, entropy functionals, on one hand and probability and power distributions and related metrics on the other. This was a significant undertaking that was a focus of the project and has been accomplished and reported in several of the publications that resulted under the present grant (see e.g., [63, 46, 8]).

2.2.1 Boltzmann Entropy and the heat equation

We consider the Boltzmann entropy

$$S := -\int \mu \log \mu$$

as an "information" potential and evaluate S along a 1-parameter family in $\mathcal{C}$, $\mu(t,x)$ or simply μ, reserving μ_t for its derivative. Integrals are with respect to the spatial variable x. The derivative of S with respect to t is

$$\frac{dS}{dt} = -\int (\mu_t \log \mu + \mu_t) = -\int (\mu_t \log \mu), \tag{8}$$

since $\int \mu = 1$. In view of the characterization of the Wasserstein norm from Equation (7), we see that the steepest gradient direction (*with respect to the Wasserstein metric*) is given by

$$\mu_t = \text{div}(\mu \nabla \log \mu) = \Delta \mu,$$

which is precisely the ***linear heat equation***. Finally, if we substitute $\mu_t = \Delta\mu$ into Equation (8), and integrate by parts, we get

$$\frac{dS}{dt} = \int \frac{\|\nabla \mu\|^2}{\mu} = \int \|\nabla \log \mu\|^2 \mu. \tag{9}$$

Hence the rate of entropy increase is given by the ***Fisher information*** *metric!*

2.2.2 Information metrics

In an analogous manner we have explored the implications of the OMT geometry on the space of power spectra of stochastic processes. For simplicity we assume herein that power spectra have the same total energy and thence are normalized to have integral 1. The principal reason why the OMT geometry is natural in the spectral domain is that the Wasserstein metric is weak*-continuous (generating the natural topology, like the Lévy-Prohorov metric[1]) and computationally tractable [23].

[1]Note that in probability "weak" often refers to "weak*". The latter terminology is the correct one from a functional analysis viewpoint.

We now let μ represent the power spectral density of a discrete-time random process, though extension to being a non-absolutely-continuous spectral measure is also possible. Thus, μ is taken to be non-negative on the unit circle – herein identified with $[0, 1)$, with $\mu(0) = \mu(1)$. The integral $\int \mu$ which represents the variance of the underlying random process is normalized to 1. It is natural to take as "information" potential the *differential entropy*

$$S_d := -\int \log \mu$$

as this relates to the variance of the optimal one-step-ahead prediction error which is simply $e^{\int \log \mu}$. We evaluate S_d along a 1-parameter family and take the derivative

$$\frac{dS_d}{dt} = -\int \frac{\mu_t}{\mu}. \tag{10}$$

The steepest gradient direction with respect to the Wasserstein metric is now given by

$$\mu_t = -\mathrm{div}(\mu\nabla\frac{1}{\mu}) = \mathrm{div}(\frac{\nabla\mu}{\mu}) = \frac{\Delta\mu}{\mu} - \frac{\|\nabla\mu\|^2}{\mu^2}. \tag{11}$$

This is yet another ***nonlinear heat equation***. We specialize to one (spatial) dimension ($\nabla\mu =: \mu_x$ is now simply the partial derivative with respect to x) and we write Equation (11) explicitly using partials with respect to this one dimension

$$\mu_t = \frac{\mu\mu_{xx} - (\mu_x)^2}{\mu^2}.$$

Upon substitution into (10), and integration by parts we obtain that

$$\frac{dS_d}{dt} = \int \frac{(\mu_x)^2}{\mu^3}. \tag{12}$$

2.2.3 Affine-invariant heat equation

It is natural to consider the general class of potentials

$$S_g = -\int f(\mu),$$

where f is a suitable differentiable increasing function. Then the exact computation given earlier shows that the corresponding gradient flow with respect to the Wasserstein metric is

$$\mu_t = \mathrm{div}(\mu\nabla f'(\mu)). \tag{13}$$

Moreover,

$$\frac{dS_g}{dt} = \int \mu\|f'(\mu)\|^2.$$

As an application, if we take $f(x) = \frac{1}{n-1}x^n,\ n \geq 0$, then (13) becomes $\mu_t = \Delta\mu^n$, which in one spatial variable reduces to

$$\mu_t = (\mu^n)_{xx}.$$

Interestingly, this can now be turned into an equation of the form

$$u_t = (u_{xx})^n \tag{14}$$

as follows. Simply apply $-\Delta^{-1}$ (i.e., the negative inverse of the Laplacian) to both sides, and set $u = -\Delta\mu$. For $n = 1/3$, we get the 1-dimensional ***affine invariant heat equation*** [61, 3] of great popularity in image processing. While this equation is known not to be derivable via any L^2-based gradient flow [50, 51], we have just shown that it can also be derived as such via the Wasserstein geometric structure. The Euclidean invariant geometric heat equation may be derived via an L^2 gradient descent flow [26].

Our main interest however is in natural metrics between distributions. To this end, we have explored several new metrics that may be derived using the framework introduced here. Indeed, following [4] again, we can devise geodesic distances analogous to *action integrals* (for "pressureless fluid flow"). Such integrals may be taken in the form:

$$\inf \int \int_0^1 \mu(t,x) h(v(t,x))\, dt\, dx \tag{15}$$

over all time-varying densities μ and vector fields v satisfying

$$\begin{aligned} \frac{\partial \mu}{\partial t} + \mathrm{div}(\mu v) &= 0, \\ \mu(0,\cdot) &= \mu_0,\ \mu(1,\cdot) = \mu_1. \end{aligned}$$

Here h is a strictly convex even function. (In [4], $h(v) = \|v\|^2/2$.) One can show that this leads to the flow

$$\mu_t = \mathrm{div}(\mu \nabla h^*(\nabla f'(\mu))), \tag{16}$$

where h^* is the Legendre transform of h. We have considered and detailed this framework and its connections with fundamental concepts of information and prediction theory in [63].

2.2.4 Unbalanced densities

General distributions (histograms, power spectra, spatio-temporal energy densities, images) may not necessarily be normalized to have the same integral. Thus, it is very important to devise appropriate metrics and theory. Our aim is to provide constructions for "interpolating" data in the form of distributions. Thus, we seek to view such in a natural metric space. The first candidate is the space of L^2-integrable functions. However, as we will note next, geodesics are simply linear intervals and fail to have a number of desirable properties [23, 33]. In particular, the "linear average" of two *unimodal* distributions is typically *bimodal*. Thus, important features are typically "written over". It is instructive for us to consider this first.

One can show that

$$d_{L^2}(\mu_0,\mu_1)^2 = \inf_{\mu,v} \int\int_0^1 |\partial_t \mu(t,x)|^2\, dt\, dx \tag{17}$$

over all time varying densities μ and vector fields v satisfying

$$\begin{aligned} \frac{\partial \mu}{\partial t} + \mathrm{div}(\mu v) &= 0, \\ \mu(0,\cdot) &= \mu_0,\ \mu(1,\cdot) = \mu_1. \end{aligned} \tag{18}$$

The optimality condition for the path is then given by

$$\partial_{tt}\mu(t,x) = 0,$$

which gives as optimal path the "interval" ($t \in [0,1]$)

$$\mu(t,x) = [\mu_1(x) - \mu_0(x)]t + \mu_0(x).$$

Our claim about bi-modality of a mix of two unimodal distributions is evident. On the other hand, OMT geodesics represent nonlinear mixing and have a considerably different character [33].

Yet, the L^2 problem may be used in conjunction with OMT in case of unbalanced mass distributions. Indeed, given the two unbalanced densities μ_0 and μ_1 it is natural to seek a distribution $\tilde{\mu}_1$ the closest density to μ_1 in the L^2 sense, which minimizes the Wasserstein distance $d_{\text{wass}}(\mu_0,\tilde{\mu}_1)^2$. The L^2 perturbation can be interpreted as "noise." One can then show that this problem amounts to minimizing

$$\inf_{\mu,v,\tilde{\mu}_1} \int\int_0^1 \mu(t,x)\|v\|^2\, dt\, dx + \alpha/2 \int |\mu_1(x) - \tilde{\mu}_1(x)|^2\, dx \tag{19}$$

over all time varying densities μ and vector fields v satisfying

$$\begin{aligned} \frac{\partial \mu}{\partial t} + \mathrm{div}(\mu v) &= 0, \\ \mu(0,\cdot) &= \mu_0,\ \mu(1,\cdot) = \tilde{\mu}_1\ . \end{aligned} \tag{20}$$

This idea has been taken further in [23, 33] where the two end points μ_0, μ_1 are allowed to be perturbed slightly into $\tilde{\mu}_0, \tilde{\mu}_1$ while the perturbation equalizes their integrals and is accounted for in the metric. This leads to a modified Monge-Kantorovich problem which is best expressed in terms of its dual. Recall that the original OMT-problem of transferring (balanced) μ_0 into μ_1 has the following dual (see e.g., [68])

$$\max_{\phi(x)+\psi(y)\le\rho(x,y)} \int \phi(x)\mu_0(x)dx + \int \psi(y)\mu_1(y)dy, \tag{21}$$

which for the case $\rho(x,y) = |x-y|$ can be shown to be

$$\max_{\|\phi\|_{\mathrm{Lip}}\le 1} \int \phi(\mu_0 - \mu_1)$$

with $\|\phi\|_{\text{Lip}} = \sup \frac{|g(x)-g(y)|}{|x-y|}$ the Lipschitz norm (see [68]). Replacing μ_0, μ_1 by $\tilde{\mu}_0, \tilde{\mu}_1$ in the OMT problem while penalizing the magnitude of the errors $\|\mu_i - \tilde{\mu}_i\|$ leads to the following metric [23]:

$$\inf_{\tilde{\mu}_0(\Omega)=\tilde{\mu}_1(\Omega)} d_1(d\tilde{\mu}_0, d\tilde{\mu}_1) + \kappa \sum_{i=1}^{2} d_{\text{TV}}(d\mu_i, d\tilde{\mu}_i). \tag{22}$$

This metric in fact "interpolates" the total variation (which is not weak* continuous) and the Wasserstein distance. The above expression has an interesting physical interpretation, where μ_i's are noisy version of the $\tilde{\mu}_i$'s and considerable practical significance, as it allows comparing distributions of unequal mass in a natural way. Further, this is weak* continuous. The dual formulation is particularly simple:

$$d_{\text{unbalanced}}(\mu_0, \mu_1) := \max_{\substack{\|\phi\|_{\text{Lip}} \le 1 \\ \|\phi\|_\infty \le c}} \int \phi(\mu_0 - \mu_1). \tag{23}$$

The constant c depends on how much penalty κ is placed on $\|\mu_i - \tilde{\mu}_i\|_2$. Geodesics of this metric retain several of the features at the end points [23, 33] but, it is also substantially more difficult to compute (distances, geodesics, etc.). Parallel work on the geometry for unbalanced mass distributions using substantially different tools and direction has been pursued in [5, 14]. Besides our interest in developing natural metrics for unbalanced mass distributions, our work exemplifies the focus in this respect: develop computational tools for (weak*) metrics between unbalanced mass distributions and, in particular, for (23). See also [47] for our recent development of a general framework and specific results under the grant for transport of matrix-valued distributions.

2.2.5 Tracking

One of the benefits of a geometric framework is the availability of geodesics and geodesic paths for linking together time-varying spectra. This is in complete analogy with the usage of approximation techniques in tracking dynamic changes in traditional system identification. Herein, spectral geodesics represent a tool for tracking changes in the spectral domain. There are several possible natural geometries. Here we continue on the one which we discussed in the previous section.

It can be shown (see [31]) that the geodesic distance between two matrix-valued power spectral density functions μ_0, μ_1 is

$$\sqrt{\frac{1}{2\pi} \int \| \log \mu_0^{-1/2} \mu_1 \mu_0^{-1/2} \|_{\text{Fr}}^2}, \tag{24}$$

with $\|\cdot\|_{\text{Fr}}$ denoting the Frobenius norm. Thus, (24) is a matricial generalization of the commonly used logarithmic deviation, and shows that the latter is in fact a meaningful geometric quantity. Further, the geodesic path between the two is

$$\mu_t = \mu_0^{1/2} (\mu_0^{-1/2} \mu_1 \mu_0^{-1/2})^t \mu_0^{1/2}, \tag{25}$$

where $t \in [0, 1]$. Likewise, using (25) we can construct geodesics between matrix-valued power spectra. Spectral geodesics represent an effective non-parametric model for nonstationarity [33]. Recent results along this line for matrix-valued geometric transport are reported in [46, 15].

2.2.6 Matricial Fisher information

Metrics between matrix-valued distributions which generalize the Fisher information have also been developed for the purposes of assessing information in the setting of quantum mechanics. Briefly, the scalar Fisher information metric is of the form $\int \frac{\delta^2}{\mu}$. The "quantum" analog, where the perturbation Δ is a matrix as is the distribution M, the Fisher information metric takes the form

$$\mathrm{trace}(\Delta \cdot \mathbb{D}_M(\Delta))$$

where $\mathbb{D}_M(\Delta)$ is a "super-operator" representing division of Δ by M. Several options for this "non-commutative" division exist. For instance, $Y = \mathbb{D}_M(\Delta)$ can be selected to satisfy $\frac{1}{2}(YM + MY) = \Delta$, but also $Y = M^{-\frac{1}{2}}\Delta M^{-\frac{1}{2}}$, or $\int_0^\infty (I + \sigma M)^{-1}\Delta(I + \sigma M)^{-1}d\sigma$ (see e.g., [56, 38], and [16, Appendix] and the references therein). There are strong connections between these metrics, which constitute a family *"non-commutative" Fisher infromation metrics*, and the "predictive geometry" we discuss in the previous section. Our relevant work that focuses on multivariable covariance statistics, under AFOSR support, is [17, 18, 19, 20] and our more recent development is detailed in [31, 48, 32, 49, 15].

2.3 Unbalanced OMT optical flow for dynamic textures

Optical flow is a computational procedure to compute the motion between a set of images, taken within a short time difference. The main idea is that the gray values of each image do not change between two images. This leads to the *optical flow constraint*

$$I_t + \vec{u} \cdot \nabla I = 0. \tag{26}$$

where I is the image and $\vec{u} = [u, v]$ is the flow field. Given two images taken in a short time interval, it is possible to solve for the optical flow field $\vec{u}$ by solving the following optimization problem

$$\min_{\vec{u}} \quad \|I_t + \vec{u} \cdot \nabla I\|^2 + \alpha R(\vec{u}) \tag{27}$$

where $R(\vec{u})$ is a regularization operator, typically chosen to be the gradient of $\vec{u}$ and α is a regularization parameter.

The underlying assumption this model is one of intensity constancy. Under this assumption an objects brightness is constant from frame to frame. This assumption holds for rigid objects with a Lambertian surface, but fails for fluid and gaseous materials. In computer vision, these are modeled by so-called *dynamic textures* (see [11]). The dynamic textures typical of smoke and fire possess intrinsic dynamics and so cannot be reliably captured by the standard optical flow method. Also, the fire/smoke region tends to flow much faster than the area around it which again may cause the model to produce erroneous results.

Our goal in this research program has been to obtain better optical flow field models for fire and smoke modeled as ***dynamic textures***. One way to do so is to base the optical flow on the physical attributes of these processes. One simple attribute is that fire and smoke tends to approximately

conserve intensity taken as a generalized mass and move the mass in an optimal way. Thus, an appropriate mathematical optical constraint is not intensity preserving but rather mass preserving. This model can be written as

$$I_t + \text{div}\, \vec{u} I = 0. \tag{28}$$

Our model for optimal flow is the one that minimizes the total energy defined as follows:

$$\inf_{\vec{u},\mu_1,\tilde{\mu}} \int_\Omega \int_0^1 \mu(x,t)\, |\vec{u}|^2\, dx\, dt + \alpha/2 \int (\tilde{\mu}_1(x) - \mu_1(x))^2 dx + \beta/2 \int_\Omega \int_0^1 (\mu_t + \text{div}\, \vec{u}\mu)^2\, dx\, dt. \tag{29}$$

subject to

$$\mu(x,0) = \mu_0(x), \quad \mu(x,1) = \tilde{\mu}_1(x).$$

Our results have been presented in several publications, see e.g., [43].

2.4 Mass Preserving Maps of Minimal Distortion

Both conformal (angle preserving) and mass preserving diffeomorphisms are of great interest in surface deformations, and therefore in image registration and template-based tracking algorithms; see [1, 30]. It is obviously very important to preserve as much geometric structure (both local and global) in the deformation map. We have considered ways of finding area preserving maps which distort shape minimally in the following sense.

Let $\mathcal{M}$ and $\mathcal{N}$ be two compact surfaces of genus 0 equipped with Riemannian metrics h and g, respectively, and let $\phi : \mathcal{M} \to \mathcal{N}$ be an area preserving map (i.e., if Ω_g and Ω_h are the area forms, then $\phi^* (\Omega_g) = \Omega_h$). Once you have ϕ there are many other area preserving maps from $\mathcal{M}$ to $\mathcal{N}$ (just compose ϕ with any other area preserving $\psi : \mathcal{N} \to \mathcal{N}$). We are interested in finding the one which has *minimal distortion.*

One possible answer is to try to find a map $\psi \circ \phi$ which minimizes the Dirichlet integral

$$\mathcal{D}[\phi] = \frac{1}{2} \int_\mathcal{M} \|D\phi\|^2\, \Omega_h. \tag{30}$$

We have already written down possible steepest descent flows which would deform an area preserving map $\phi : \mathcal{M} \to \mathcal{N}$ within the class of area preserving maps to a critical point (most likely a local minimum) of $\mathcal{D}$. In the classical Dirichlet problem, one derives a conformal map as the minimizer of such an integral. In the present case, we are minimizing the Dirichlet integral over a more restrictive class of maps to get area preservation with the least angular distortion.

We have considered gradient descent flows for minimizing (30) including existence and uniqueness. Indeed, even though short time existence does not follow from the standard theory on parabolic systems, Hamilton's approach using the Nash-Moser implicit function theorem is applicable, and leads to the soughtafter result.

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Abstract

We developed several new directions in the theory and applications of Optimal Mass Transport (OMT). OMT has its origins in civil engineering (Monge 1781) and economics (Kantorovich 1942), but in recent years has increasingly impacted a large number of other fields (probability theory, partial differential equations, physics, meteorology). We have addressed computational aspects of the problem and the need for further expanding the arsenal of computational tools. We considered a wide range of generalizations and insights for the purpose of tackling problems of AFOSR interest. These include matrix-valued statistics and fusion of information, optical flow, controlled active vision, tracking and dynamic textures.

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Do you have any discoveries, inventions, or patent disclosures to report for this period?

No

Please describe and include any notable dates

Do you plan to pursue a claim for personal or organizational intellectual property?

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None

Change in AFOSR Program Officer, if any:

Dr. Frederick A. Leve AFOSR Dynamics and Control, Phone (703) 696-7305

Extensions granted or milestones slipped, if any:

Extended granted to September 15, 2016.

AFOSR LRIR Number

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Laboratory Task Manager

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